Power Series Method for the Solution of Nonlinear Volterra Integral Equations: A veritable tool in Engineering Mathematics

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ABSTRACT

In engeering modeling, simulation and anlysis, mathematical formulations that always result into differential, integral even integro-diffrential equations have remained fundamental and valid tools while the solution to the resultant equations will be needed to interprete and explain the behavioural partern of the phenomena under study. Integral equations as part of these resulting equations have different applications in engeering such as determination of potentials, system identification, spectroscopy and seismic travel time. This work considered the solution to nonlinear Volterra Integral Equations(VIE) using power series approach. The degenerate kernel is considered with numerical examples in the methodology for the work. The equation is resolved into system of nonlinear equations and solved to obtain the unknown coefficients which will explain the behaviour of the serial solution. The results show that the power series method is very useful, especially when it is difficult to obtain the analytic solution, easy to compute and very effective for handling problems.

KEYWORDS: Degenerate kernel, Power series, Volterra integral equation,.

1. INTRODUCTION

The remarkable advancement in different field of engineering, science and technology and emerging applications in mathematical sciences, mathematical physics and chemical reactions including stereology have made the study of Volterra integral equations becomes essential and play prominent roles in the modeling and solution of these phenomena(Al-Jawary & Shehan; 2015; Linz, 1985; Mangey & Paulo, 2019). It is not an understatement to re-emphasis here that to have an exhaustive understanding of subjects like fluid dynamics, numerical analysis, waves and electromagnetic, chemistry, physics, statistics, mathematical biology, aerodynamics, electricity and even financial engineering, the knowledge of determining the solution of integral is absolutely necessary(Aggarwal et al, 2020) Finding and interpreting the solutions of these integral equations are therefore central parts of applied mathematics and engineering and a thorough understanding of integral equations will be essential for the engineering practitioners. In formation processes, these equations can either be linear or nonlinear in type(Jerri,1985), (Abdul, 1999), (Shanti, 2007), (Wazwaz, 2011) . It is also of importance note that the types or nature of the kernels determine the behavior of the equations. Chen and Lin (2008) discussed a new reproducing kernel space of $Au = f$ where the image space is bounded and it guaranteed existence of exact solution. To obtain solution to these classes of equations, a very huge approaches have been adopted.

Tricomi (1982) and Wazwaz (2011) used classical method of successive approximations to solve nonlinear Volterra integral equations while implicit linear collocation approach was applied by (Brunner, 1992) and (Tahmasbi and Fard, 2008) used direct power series method. Mohand transform, Abooh transform, Mahgoub transform and host of others have been used to obtain mainly analytic solutions to VIE. Aggarwal et al, (2018) used the transformation techniques to obtain solution for second kind while other analytical methods such as Laplace, Maghoub etc have also been applied. Bani issa et al, (2019) lately applied semi-analytic numerical techniques to obtain solution to nonlinear VIE.

The aim of this work is to recover power series from Taylor series and apply it to determine the numerical solution of nonlinear Volterra integral equations by series expansion. Generally, nonlinear VIE is of the form

$$y(\eta) = \lambda\int_0^\eta K(\eta,s)G(y(s))ds + f(\eta) \quad \eta \in [0,\Theta], \tag{1}$$

which is referred to as nonlinear VIE of second kind. The function $G(y(\eta))$ is a nonlinear function of $y(\eta)$ such as $y^n(\eta) \quad n>1, \; Sin(y(\eta))$ and $e^{u(\eta)}$ (Wazwaz, 2011). The unknown function $y(\eta)$ is to be determined while $f(\eta)$ and $K(\eta,s)$ are given continuous functions.

If $f(\eta)=0$, (1) will become of the form

$$y(\eta) = \lambda\int_0^\eta K(\eta,s)G(y(s))ds \quad \eta \in [0,\Theta] \tag{2}$$

and is referred to as nonlinear first kind VIE (Abdul, 1999), (Agbolade and Anake, 2017), Rahman, 2007). The equation (1) can also be described as linear first kind or linear second kind depending on the analytic function $G(y(s))$. Linear VIE of both first and second kinds have the forms

$$y(\eta) = \lambda\int_0^\eta K(\eta,s)G(y(s))ds \quad \eta \in [0,\Theta] \tag{3}$$

and

$$y(\eta) = \lambda\int_0^\eta K(\eta,s)y(s)ds + f(\eta) \quad \eta \in [0,\Theta] \tag{4}$$

respectively.

2 FREQUENTLY USED POWER SERIES IN ENGINEERING AND MATHEMATICAL SCIENCES

In most of Engineering and Mathematical Sciences, series in general such as Taylor, Fourier, Frobenius, Legendre are very important considering their frequent applications in solving various types of Engineering and Mathematical modeling problems. Among the most common and frequently applied are

$$e^{\eta} = \left(1+\eta+\frac{\eta^2}{2!}+\frac{\eta^3}{3!}+\frac{\eta^4}{4!}+\frac{\eta^5}{5!}+...\right)$$

$$e^{-\eta} = \left(1-\eta+\frac{\eta^2}{2!}-\frac{\eta^3}{3!}+\frac{\eta^4}{4!}-\frac{\eta^5}{5!}+...\right)$$

$$e^{k\eta} = \left(1+k\eta+\frac{(k\eta)^2}{2!}+\frac{(k\eta)^3}{3!}+\frac{(k\eta)^4}{4!}+\frac{(k\eta)^5}{5!}+...\right)$$

$$e^{-k\eta} = \left(1-k\eta+\frac{(k\eta)^2}{2!}-\frac{(k\eta)^3}{3!}+\frac{(k\eta)^4}{4!}-\frac{(k\eta)^5}{5!}+...\right)$$

$$Sin\eta = \left(\eta-\frac{\eta^3}{3!}+\frac{\eta^5}{5!}-\frac{\eta^7}{7!}+\frac{\eta^9}{9!}+...\right)$$

$$Cos\eta = \left(1-\frac{\eta^2}{2!}+\frac{\eta^4}{4!}-\frac{\eta^6}{6!}+\frac{\eta^8}{8!}+...\right)$$

$$Tan\eta = \left(\eta+\frac{\eta^3}{3!}+\frac{2\eta^5}{15}+...\right)$$

$$Sinh^{-1}\eta = \left(\eta + \frac{\eta^3}{3!} + \frac{\eta^5}{5!} + \frac{\eta^7}{7!} + \frac{\eta^9}{9!} + ... \right)$$

$$Cosh^{-1}\eta = \left(1 + \frac{\eta^2}{2!} + \frac{\eta^4}{4!} + \frac{\eta^6}{6!} + \frac{\eta^8}{8!} + + ... \right)$$

$$Sin^{-1}\eta = \left(\eta + \frac{1}{2}\left(\frac{\eta^3}{3!} \right) + \frac{1.2}{3.4}\left(\frac{\eta^5}{5!} \right) + \frac{1.2.3}{3.4.5}\left(\frac{\eta^7}{7!} \right) + ..., \quad \eta^2 < 1 \right)$$

$$Log(1+\eta) = \left(\eta - \frac{\eta^2}{2!} + \frac{\eta^3}{3!} - \frac{\eta^4}{4!} + \frac{\eta^5}{5!} - \frac{\eta^6}{6!} + ..., -1 < \eta \le 1 \right)$$

$$Log(1-\eta) = \left(-\eta - \frac{\eta^2}{2!} - \frac{\eta^3}{3!} - \frac{\eta^4}{4!} + - \frac{\eta^5}{5!} - \frac{\eta^6}{6!} + ..., -1 \le \eta < 1 \right)$$

$$\frac{1}{(1-\eta)} = \left(1 + \eta + \eta^2 + \eta^3 + \eta^4 + \eta^5 + ..., |\eta| < 1 \right)$$

$$\frac{1}{(1+\eta)} = \left(1 + \eta - \eta^2 + \eta^3 - \eta^4 + \eta^5 - ..., |\eta| < 1 \right)$$

$$\frac{1}{(1-\eta)^2} = \left(1 + 2\eta + 3\eta^2 + 4\eta^3 + 5\eta^4 + \eta^5 + ..., |\eta| < 1 \right)$$

$$\frac{1}{(1-\eta)^3} = \left(1 + 3\eta + 6\eta^2 + 10\eta^3 + 20\eta^4 + ..., |\eta| < 1 \right)$$

$$\frac{1}{(1+\eta)^{\frac{1}{2}}} = \left(1 + \frac{\eta}{2} + \frac{\eta^2}{8} + \frac{\eta^3}{16} + ..., |\eta| < 1 \right)$$

$$\frac{1}{(1+\eta)^{-\frac{1}{2}}} = \left(1 - \frac{\eta}{2} + \frac{\eta^2}{8} - \frac{\eta^3}{16} + ..., |\eta| < 1 \right)$$

3. RESEARCH METHODOLOGY

3.1 Introduction

The general form of nonlinear Volterra integral equation, (1.1) is considered here.

$$y(\eta) = \lambda \int_0^{\eta} K(\eta, s) G(y(s)) ds + f(\eta) \quad \eta \in [0, \Theta]. \tag{5}$$

where $K(\eta, s)$ and $f(\eta)$ are given real valued functions, and $G(y(\eta)$ is a nonlinear continuous function of which the unknown function $y(\eta)$ is to be determined.The reviewed series of functions stated in Section 2 were generated from Taylor's series computations, hence we want to assume an approximate solution to the VIE by considering the power series which is achieved by evaluating Taylor's series (Simmons and Krantz, 2007)

$$y(\eta) = \sum_{n=0}^{\infty} \alpha_n (\eta - a)^n$$

at $\eta = 0$.

For the existence of the solution for this equation, we will state the conditions for existence as in (Kreyszig, 1979), (Wazwaz, 2011).

The conditions are

(i) The function $f(\eta)$ is integrable and bounded in $[a,b]$.

(ii) The function $f((\eta)$ must satisfy the Lipschitz condition in the interval (a,b) such that

$$|f(\eta)-f(y)|<\rho|\eta-y|.$$

(iii) The function $F(\eta,s,y(s))$ is integrable and bounded, $|F(\eta,s,y(s))|<K$ in $[a,b]$. must satisfy Lipschitz condition

$$|f(\eta)-f(y)|<\rho|\eta-y|.$$

(iv) The function $F(\eta,s,y(s))$ must satisfy Lipschitz condition

$$|F(\eta,s,z)-F(\eta,s,z')|<M|z-z'|.$$

3.2 Scheme Formulation

We consider an approximation to the solution of (2) in the form of truncated version

$$y_N(\eta)=\sum_{n=0}^{N}\alpha_n\eta^n. \tag{6}$$

Substituting (6) into (5) yields

$$\sum_{n=0}^{N}\alpha_n\eta^n = f(\eta)+\lambda\int_0^\eta K(\eta,s)G\left(\sum_{n=0}^{N}\alpha_n s^n\right)ds \tag{7}$$

Since the function $f(\eta)$ is analytic, it can also be represented in series form.

Therefore (7) becomes

$$\sum_{n=0}^{N}\alpha_n\eta^n = T(f(\eta))+\lambda\int_0^\eta K(\eta,s)G\left(\sum_{n=0}^{N}\alpha_n s^n\right)ds. \tag{8}$$

Suppose the nonlinear function $G(y(s))$ is a power function of $y(s)^p$ then (3) can be expressed as

$$y(\eta)=\lambda\int_0^\eta K(\eta,s)G(y(s)^p)ds+f(\eta)\quad \eta\in[0,\Theta]. \tag{9}$$

For $p=2$, (9) becomes a nonlinear Hammerstein-type integral equation (Tahmasbi and Fard, 2008) which now is

$$\sum_{n=0}^{N}\alpha_n\eta^n = T(f(\eta))+\lambda\int_0^\eta K(\eta,s)G(\left(\sum_{n=0}^{N}\alpha_n s^n\right)^p)ds \tag{10}$$

Collecting like terms

$$\sum_{n=0}^{N}\alpha_n\eta^n-\lambda\int_0^\eta K(\eta,s)G\left(\sum_{n=0}^{N}a_n s^n\right)^p ds = T(f(\eta)) \tag{11}$$

and written in compact form, we have that

$$T(\eta)A-\vartheta(\eta)A^k = g(\eta) \tag{12}$$

where

$$T(\eta)=\sum_{n=0}^{N}\alpha_n\eta^n,$$

$$\vartheta(\eta)=\lambda\int_0^\eta K(\eta,s)G(\left(\sum_{n=0}^{N}\alpha_n s^n\right)^p)ds,$$

and

$$A=[\alpha_0 \quad \alpha_1 \quad \cdots \quad]^T.$$

The equation (12) is now a system of nonlinear algebraic equations which can be solved using any standard software packages.

Subsequently, the determined unknown coefficients in (12) are substituted into (6) to obtain the desired approximate result given by

$$y_N=\alpha_0+\alpha_1\eta+\alpha_2\eta^2+\ldots+\alpha_N\eta^N. \tag{13}$$

This approximate solution sometimes can approximate to the exact solution while some other complicated problems will only be given in approximated solution.

4. NUMERICAL EXPERIMENTS AND RESULTS

For efficiency of the derived scheme, some experimental examples were presented with both first kind and second kind Volterra integral equations.

4.1 Numerical Experiments

Example 4.1 *Consider the first kind nonlinear VIE*

$$\frac{1}{4}e^{2\eta}-\frac{1}{2}\eta-\frac{1}{4}=\int_0^\eta(\eta-s)(y(s))ds$$

and the exact solution is y(x)=e^-

The scheme is applied to the problem.

We assume the approximate solution (6) to the problem to obtain

$$T(f(\eta))=\int_0^\eta(\eta-s)(y(s))^2d\eta$$

$$f(\eta)=\frac{1}{4}e^{2\eta}-\frac{1}{2}\eta-\frac{1}{4} \tag{14}$$

and (14) becomes

$$\frac{1}{4}e^{2\eta}-\frac{1}{2}\eta-\frac{1}{4}=\int_0^\eta(\eta-s)(y(s))^2d\eta \tag{15}$$

Substituting the series for $e^{2\eta}$ from Section 2.0 into (15), then (15) can now be written as

$$\frac{1}{4}\left[1+\frac{2\eta}{1!}+\frac{4\eta^2}{2!}+\frac{8\eta^3}{3!}+\frac{16\eta^4}{4!}+\frac{32\eta^5}{5!}+\frac{64\eta^6}{6!}+\ldots\right]-\frac{1}{2}\eta-\frac{1}{4}$$

$$=\int_0^\eta(\eta-s)\left(\sum_{n=0}^{N}\alpha_n s^n\right)^2d\eta \tag{16}$$

and upon expansion, we have that

$$\frac{1}{4}\left[1+\frac{2\eta}{1!}+\frac{4\eta^2}{2!}+\frac{8\eta^3}{3!}+\frac{16\eta^4}{4!}+\frac{32\eta^5}{5!}+\frac{64\eta^6}{6!}+...\right]-\frac{1}{2}\eta-\frac{1}{4}$$

$$=\int_0^\eta(\eta-s)(\alpha_0+\alpha_1 s+\alpha_2 s^2+\alpha_3 s^3+\alpha_4 s^4+\alpha_5 s^5+...)^2 ds \quad (17)$$

which is now resolved into

$$\left[\frac{\eta^2}{2}+\frac{\eta^3}{3}+\frac{\eta^4}{6}+\frac{\eta^5}{15}+\frac{\eta^6}{45}+...\right]$$

$$=\int_0^\eta(\eta-s)(\alpha_0+\alpha_1 s+\alpha_2 s^2+\alpha_3 s^3+\alpha_4 s^4+\alpha_5 s^5+...)^2 ds \quad (18)$$

Evaluating (18), we obtain

$$\left[\frac{\eta^2}{2}+\frac{\eta^3}{3}+\frac{\eta^4}{6}+\frac{\eta^5}{15}+\frac{\eta^6}{45}+...\right]=\int_0^\eta(\eta-s)(\alpha_0^2+2\alpha_0\alpha_1\eta+(\alpha_1^2+2\alpha_0\alpha_2)\eta^2+$$

$$+(2\alpha_1\alpha_2+2\alpha_0\alpha_3)\eta^3+(\alpha_2^2+2\alpha_1\alpha_2+2\alpha_0\alpha_3)\eta^4+(2\alpha_2\alpha_3+2\alpha_1\alpha_2+2\alpha_0\alpha_5)\eta^5+...)ds \quad (19)$$

and after simplification, (19) results into a nonlinear polynomial.

$$\left[\frac{\eta^2}{2}+\frac{\eta^3}{3}+\frac{\eta^4}{6}+\frac{\eta^5}{15}+\frac{\eta^6}{45}+...\right]=\eta\left[\begin{array}{l}\alpha_0^2+\alpha_0\alpha_1\eta^2+(\alpha_1^2+2\alpha_0\alpha_2)\frac{\eta^3}{3}+(2\alpha_1\alpha_2+2\alpha_0\alpha_3)\frac{\eta^4}{4}+\\(\alpha_2^2+2\alpha_1\alpha_2+2\alpha_0\alpha_3)\frac{\eta^5}{5}+(2\alpha_2\alpha_3+2\alpha_1\alpha_2+2\alpha_0\alpha_5)\frac{\eta^6}{6}+...\end{array}\right]-$$

$$[\alpha_0^2\frac{\eta^2}{2}+2\alpha_0\alpha_2\frac{\eta^3}{3}+(\alpha_1^2+2\alpha_0\alpha_2)\frac{\eta^4}{4}+(2\alpha_1\alpha_2+2\alpha_0\alpha_3)\frac{\eta^5}{5}+(\alpha_2^2+2\alpha_1\alpha_3+2\alpha_0\alpha_4)\frac{\eta^6}{6}+....] \quad (20)$$

Simply further by equating few of the coefficients on both sides of (20) yields a system of nonlinear algebraic equations

$$\frac{1}{2}=\alpha_0^2-\frac{\alpha_0^2}{2}$$

$$\frac{1}{3}=\alpha_0\alpha_1-\frac{2\alpha_0\alpha_1}{3}$$

$$\frac{1}{6}=\frac{\alpha_1^2+2\alpha_0\alpha_2}{3}-\frac{\alpha_1^2+2\alpha_0\alpha_2}{4}$$

$$\frac{1}{15}=\frac{2\alpha_1\alpha_2+2\alpha_0\alpha_3}{4}-\frac{2\alpha_1\alpha_2+2\alpha_0\alpha_3}{5}$$

$$\frac{1}{45}=\frac{\alpha_2^2+2\alpha_1\alpha_3+2\alpha_0\alpha_4}{5}\frac{\alpha_2^2+2\alpha_1\alpha_3+2\alpha_0\alpha_4}{6}$$

which are solved to obtain

$$\begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix}^T = \begin{bmatrix}1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24}\end{bmatrix}^T \tag{21}$$

and

$$\begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix}^T = \begin{bmatrix}-1 & -1 & -\frac{1}{2} & -\frac{1}{6} & -\frac{1}{24}\end{bmatrix}^T \tag{22}$$

respectively. The coefficients are now substituted into the assumed approximate solution (3) to obtain

$$\begin{aligned} y(\eta) &= 1+\eta+\frac{1}{2}\eta^2+\frac{1}{6}\eta^3+\frac{1}{24}\eta^4+\ldots \\ &= e^{\eta} \\ y(\eta) &= -1+(-\eta)+(-\frac{1}{2}\eta^2)+(-\frac{1}{6}\eta^3)+(-\frac{1}{24}\eta^4)+\ldots \\ &= e^{-\eta} \end{aligned} \tag{23}$$

Example 4.2 *Consider again nonlinear Volterra Integral equation of second kind*

$$y(\eta) = f(\eta) + \int_0^{\eta} (\eta - s)^2 (y(s)^2) ds$$

where the exact solution is y()=e^

Source: (Wazwaz, 2011)

The scheme is applied to the problem.

We assume the approximate solution (6) to the problem to obtain

$$\sum_{n=0}^{N} \alpha_n \eta^n = f(\eta) + \int_0^{\eta} (\eta - s)^2 \left(\sum_{n=0}^{N} \alpha_n s^n \right)^2 ds$$

$$f(\eta) = \frac{1}{4} + \frac{1}{2}\eta + \frac{1}{2}\eta^2 + e^{\eta} - \frac{1}{4}e^{2\eta} \tag{24}$$

We evaluate $f(\eta)$ by substituting the series polynomials for e^{η} and $e^{2\eta}$ respectively to obtain

$$f(\eta) = \frac{1}{4} + \frac{1}{2}\eta + \frac{1}{2}\left[1+\eta+\frac{\eta^2}{2!}+\frac{\eta^3}{3!}+\frac{\eta^4}{4!}+\frac{\eta^5}{5!}+\ldots\right] - \frac{1}{4}\left[1+2\eta+\frac{(2\eta)^2}{2!}+\frac{(2\eta)^3}{3!}+\frac{(2\eta)^4}{4!}+\frac{(2\eta)^5}{5!}+\ldots\right] \tag{25}$$

hence

$$f(\eta) = 1+\eta+\frac{1}{2}\eta^2-\frac{1}{6}\eta^3-\frac{1}{8}\eta^4-\frac{7}{120}\eta^5+\ldots \tag{26}$$

and (4.11) now becomes

$$\sum_{n=0}^{N} \alpha_n \eta^n - \int_0^{\eta} (\eta - s)^2 \left(\sum_{n=0}^{N} \alpha_n s^n \right)^2 ds = 1+\eta+\frac{1}{2}\eta^2-\frac{1}{6}\eta^3-\frac{1}{8}\eta^4-\frac{7}{120}\eta^5+\ldots \tag{27}$$

Evaluating (27) yields a nonlinear polynomial

$$\ldots - \frac{1}{858}\eta^{13}\alpha_5^2 - \frac{1}{330}\eta^{12}\alpha_4\alpha_5 - \left(\frac{2}{495}\alpha_3\alpha_5 - \frac{2}{495}\alpha_4^2\right)\eta^{11} -$$
$$\left(\frac{1}{180}\alpha_2\alpha_5 + \frac{1}{180}\alpha_3\alpha_4\right)\eta^{10} - \left(\frac{1}{126}\alpha_1\alpha_5 + \frac{1}{126}\alpha_2\alpha_4 + \frac{1}{252}\alpha_3^2\right)\eta^9$$
$$-\left(\frac{1}{84}\alpha_0\alpha_5 + \frac{1}{84}\alpha_1\alpha_4 + \frac{1}{126}\alpha_2\alpha_3\right)\eta^8 - \left(\frac{2}{105}\alpha_0\alpha_4 + \frac{2}{105}\alpha_1\alpha_3 + \frac{1}{105}\alpha_2^2\right)\eta^7 - \left(\frac{1}{30}\alpha_0\alpha_3 - \frac{1}{30}\alpha_1\alpha_2\right)\eta^6 -$$
$$\left(\frac{1}{15}\alpha_0\alpha_2 - \frac{1}{30}\alpha_1^2 + \alpha_5\right)\eta^5 - \left(\frac{1}{6}\alpha_0\alpha_1 - \alpha_4\right)\eta^4 - (\frac{1}{3}\alpha_0^2 - \alpha_3)\eta^3 + \alpha_2\eta^2 + \alpha_1\eta + \alpha_0 =$$
$$1 + \eta + \frac{1}{2}\eta^2 - \frac{1}{6}\eta^3 - \frac{1}{8}\eta^4 - \frac{7}{120}\eta^5 + \ldots$$

Equating few of the terms with variable η in the nonlinear polynomial gives

$$\begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \\ -\frac{1}{6} \\ -\frac{1}{8} \\ -\frac{7}{120} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 - \frac{1}{3}\alpha_0^2 \\ \alpha_4 - \frac{1}{6}\alpha_0\alpha_1 \\ \alpha_5 - \frac{1}{15}\alpha_0\alpha_2 - \frac{1}{30}\alpha_1^2) \end{bmatrix} \quad (28)$$

The system of nonlinear algebraic equations (28) is solved to obtain the values of the coefficients

$$\begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5\end{bmatrix}^T = \begin{bmatrix}1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120}\end{bmatrix}^T \quad (29)$$

The coefficients are now substituted into (6) to yield

$$\begin{aligned} y_N &= 1 + \eta + \frac{1}{2}\eta^2 + \frac{1}{6}\eta^3 + \frac{1}{24}\eta^4 + \frac{1}{120}\eta^5 + \ldots \\ &= e^{\eta} \end{aligned} \quad (30)$$

which is the exact solution.

4.2 Conclusion

Power series is popular among the scientists. Application of The series is discussed in this work for determining the solution of nonlinear Volterra integral equations. Numerical examples of both first and second kinds are considered and the results show that the methodology is effective in obtaining the solutions. The computations are straight forward and moderate.

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